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Comparison of MHD Stability Results Obtained with the BETA 3D and HERA 2D Codes*

O. BETANCOURT

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

F. HERRNEGGER, P. MERKEL, AND J. NÜHRENBERG

Max-Planck-Institut für Plasmaphysik, EURATOM Association, D-8046 Garching b. Munich, West Germany

AND

R. GRUBER AND F. TROYON

Centre de Recherches en Physique des Plasmas, EURATOM Association, École Polytechnique Féderal de Lausanne, Lausanne, Switzerland

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Comparison of MHD stability results obtained for helically symmetric equilibria with two codes, HERA and BETA, showed excellent agreement in growth rates and marginal stability parameters. While HERA is the helical version of the ERATO linearized 2D stability code, BETA obtains nonlinear 3D equilibria and their (nonlinear) stability properties. The cases studied comprise internal, i.e., fixed plasma boundary, resonant and nonresonant modes in low-shear equilibria with intermediate and small β values.

1. INTRODUCTION

The assessment of the MHD stability properties [1] of stellarator [2] equilibria is a challenging 3D computational task [3–5]. Although a significant amount of results [4, 6–8] was obtained for truly toroidal stellarators, here understood as ideal MHD equilibria with vanishing net toroidal current on each magnetic surface, independent tests of such results are as desirable as they are hard to come by [9]. Helically symmetric equilibria provide the possibility of rather relevant tests since they can be chosen net-current-free so that purely pressure-driven ideal MHD modes can be studied. The HERA code, which investigates the linearized 2D stability problem with

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the help of an eigenvalue analysis based on the Ritz–Galerkin method, recently became operational [10]. The BETA nonlinear energy-minimizing 3D code [4, 8] can of course be applied to the case of helical symmetry, so that quantitative comparisons can be obtained.
In the following two classes of helically symmetric equilibria with straight magnetic axes are analyzed with respect to the stability of selected modes. The equilibrium sequences are defined in Section 2, and stability results are presented in Section 3.

2. Two Classes of Equilibria

We describe two classes of helically symmetric $l=2$ equilibria as obtained with BETA and the HASE quasianalytical 2D equilibrium code [11].

2.1. *Class* 1. Here, we consider $l=2$ equilibria with fixed rotational transform on axis $\iota_0 = 0.0275$ per field period, corresponding to a half-axis ratio of the elliptical cross section near the axis $e = 1.404307$, and fixed aspect ratio A of the field period ($A = L_p/2\pi a_{\text{plasma}}$) of $A = 1.7$. For the shape of the flux surfaces see Fig. 1. The parameter which is varied in class 1 is β,

$$0.05 \leqslant \langle\beta\rangle \leqslant 0.15,$$

where the average β value is defined by

$$\langle\beta\rangle = 2\int p\,dV \Big/ \iiint \mathbf{B}^2\,d^3\tau \qquad \left(V = \iiint d^3\tau\right)$$

Nearly parabolic (in distance from the axis) pressure profiles are used; see Table I. In HASE, the condition of vanishing net longitudinal current on each surface is satisfied by adjusting the arbitrarily prescribable profiles. The half-axis ratio of the elliptical

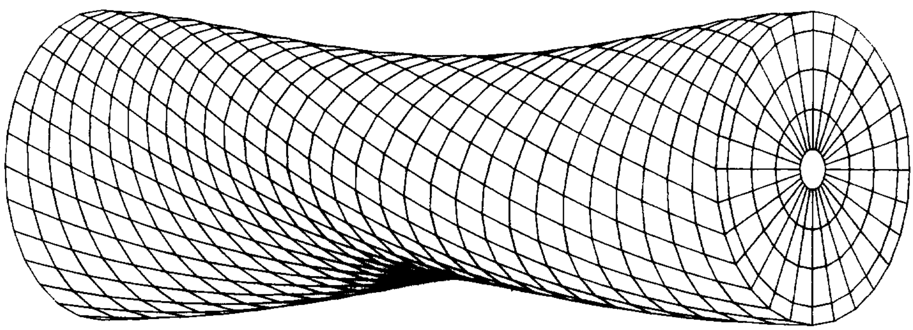

FIG. 1. One field period of the equilibria of class 1 as obtained with HASE. Aspect ratio $A = 1.7$, rotational transform per field period on axis $\iota_0 = 0.0275$.

TABLE I

$\langle\beta\rangle$	p_0	p_1	Δ_2
0.1529	1.74825	−2.86	0.157
0.1272	1.46475	−1.98	0.170
0.1018	1.18125	−1.262	0.162

Note. p_0 and p_1 are the constants in the pressure profile $p' = -p_0 + p_1 T$ used in HASE; Δ_2 is the ellipticity of the contour of the boundary in BETA; $\Delta_2 = (e_b - 1)/(e_b + 1)$. $\iota_0 = 0.0275$ per field period ($e_0 = 1.404307$) for all equilibria of class 1.

contours of the plasma boundary varies slightly with β and is used as input for BETA; see Table I. BETA, when solving for vanishing net longitudinal current, then obtains, in the limit zero mesh size, the same ι-profiles.

2.2. *Class* 2. Here, we consider $l = 2$ equilibria with fixed β, $\langle\beta\rangle = 0.03$, approximately parabolic pressure profile, and $A = 1.51$. The parameter which varies in class 2 is ι:

$$0.06 < \iota_0 < 0.12.$$

The basic sequence was obtained with BETA by varying Δ_2; see Table II. An equivalent sequence was obtained with HASE by appropriate choices of e (corresponding to ι_0) and the fourth-order fourth-harmonic shape parameter S_{44} (for

TABLE II

Parameters of the Equilibria of Class 2 as Obtained with BETA

Δ_2	ι_0	ι_E	$5\iota_0$	$5\iota_E$
0.385	0.116	0.1358	0.58	0.679
0.355	0.1004	0.118	0.502	0.590
0.340	0.093	0.1094	0.465	0.547
0.325	0.0856	0.1008	0.428	0.504
0.295	0.072	0.0842	0.36	0.421

Note. ι_0 and ι_E are the rotational transforms at the centre and the edge, respectively. $\langle\beta\rangle = 0.03$ and $A = 1.51$ for all equilibria of class 2.

TABLE III

Parameters of the Equilibria of Class 2 as Obtained with HASE

e	$5\iota_0$	S_{44}	p_0	p_1	$5\iota_E$	Δ_2
2.08433	0.55	0.166	0.3315	0.066	0.649	0.3753
2.00000	0.50	0.15	0.3240	0.050	0.590	0.3552
1.88169	0.428	0.127	0.3166	0.026	0.504	0.3255
1.836412	0.400	0.118	0.3133	0.017	0.471	0.3134
1.79621	0.375	0.110	0.3103	0.009	0.441	0.3025

Note. e is the half-axis ratio of the elliptical cross section at the centre, and S_{44} is a fourth-order shape parameter [11].

details of notation see [11]) (corresponding to ι_E), which resulted, consistently, in purely elliptical plasma boundaries; see Table III. For the rotational transform profiles see Fig. 2; for the shape of the flux surfaces see Fig. 3.

3. Stability Results

The equilibria of class 1 were analyzed with respect to the nonresonant fixed-boundary $m = 1$, $n = 0$ mode, while the nearly resonant fixed-boundary $m = 2$, $n = 1$ mode was investigated in a topological torus consisting of five periods of the equilibria of class 2. Here, $2m$ and $2n$ are the poloidal and toroidal node numbers, respectively.

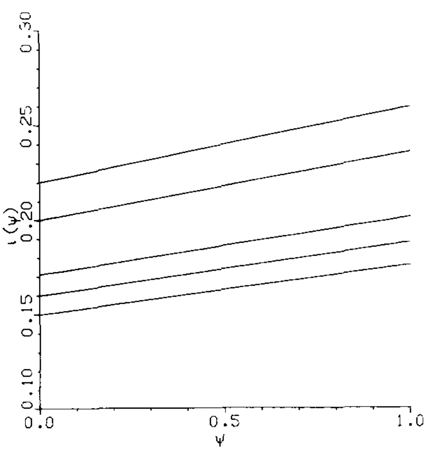

FIG. 2. Rotational transform profiles for the equilibria of class 2.

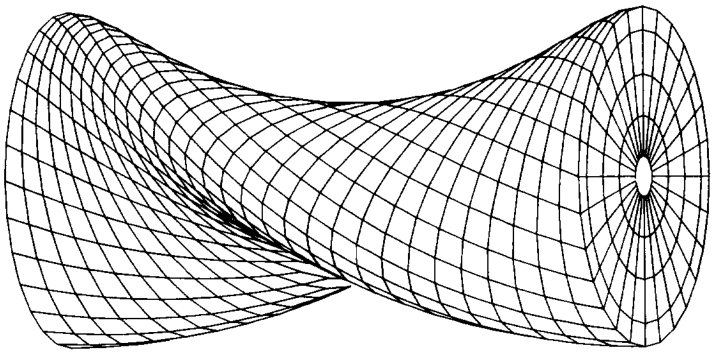

FIG 3. One field period of an equilibrium of class 2 as obtained with HASE. Aspect ratio $A = 1.51$ and $\iota_0 = 0.11$.

In helical symmetry, modes can be characterized by their longitudinal wavenumber k and their poloidal node number $2m$, so that a Fourier decomposition obtains which is exact with respect to z and approximate with respect to ζ, where $\zeta = \varphi - hz$ (r, φ, z are cylindrical coordinates and $2\pi/h$ is the helical period),

$$\xi \approx e^{ikz} e^{im\zeta}.$$

Characterizing modes in a topological torus of N helical periods ($2N$ field periods in the cases considered here) by their poloidal and toroidal node numbers

$$\xi \approx e^{i(nv + mu)},$$

where v and u are toroidal and poloidal parameters, respectively, we find

$$k/h = n/N + m,$$

which defines the k-value, which, in HERA, is characteristic of a mode with given m and n.

In BETA, δW is obtained by a secondary minimization [8] (subsequent to the minimization leading to equilibrium), in which the eigenfunction is constrained to lie on a hyperplane normal to a given test displacement defined by δr_0, δz_0 perturbing the magnetic axis and δR, $\delta\psi$ perturbing the plama region; for details of notation see [8]. For the low-shear equilibria considered here, a test displacement which models the expected mode in a simple way is sufficient. The normalized test displacements are

$$\delta r_0 = 1,$$

$$\delta z_0 = 0,$$

$$\delta R = s^{1/2}(1 - s)\cos u,$$

$$\delta\psi = -\tfrac{1}{2}(1 - 3s)\sin u$$

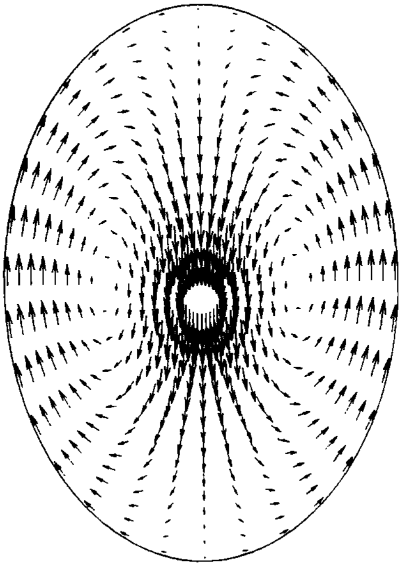

FIG. 4. The fixed-boundary $m = 1$, $n = 0$ mode occurring in equilibria of class 1.

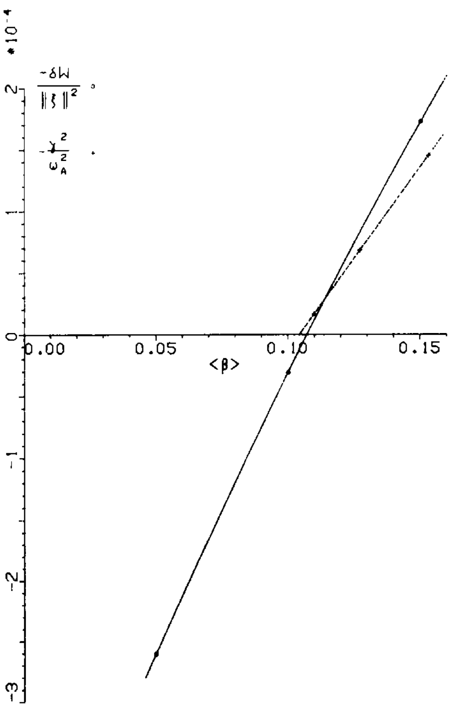

FIG. 5. Normalized eigenvalues (extrapolated to zero mesh size) of the $m = 1$, $n = 0$ mode occurring in class 1 as obtained with BETA ($-\delta W/\|\xi\|^2$) and HERA ($-\gamma^2/\omega_A^2$). The norm $\|\xi\|$ is defined by $\|\xi\|^2 = \frac{1}{2}\iiint (p/B^2)(\delta\psi\nabla s - \delta s\nabla\psi)^2\, d^3\tau$, $\rho = p$. ω_A is an Alfven frequency defined by $\omega_A^2 = B^2/4\pi\rho(0)\, a^2_{\text{plasma}}$, where the density ρ is assumed to be proportional to the pressure $p \propto \rho$.

for the $m = 1$, $n = 0$ mode and

$$\delta r_0 = \delta z_0 = 0,$$
$$\delta R = 2s^{1/2}(1-s)\cos(2u - v),$$
$$\delta\psi = -(1-2s)\sin(2u - v)$$

for the $m = 2$, $n = 1$ mode. Here $\mathbf{B} = \nabla s \times \nabla\psi$, with s the toroidal flux which scales like radius squared.

3.1. *Results for the m = 1, n = 0 mode.* The structure of the $m = 1$, $n = 0$ mode is shown in Fig. 4. Normalized eigenvalues as obtained with HERA are shown in Fig. 5 and indicate a marginally stable point with respect to the $m = 1$, $n = 0$ mode of

$$\langle\beta\rangle_{cr} = 10.5\%.$$

Normalized eigenvalues as obtained with BETA are also shown in Fig. 5 and indicate a marginal point

$$\langle\beta\rangle_{cr} = 10.7\%.$$

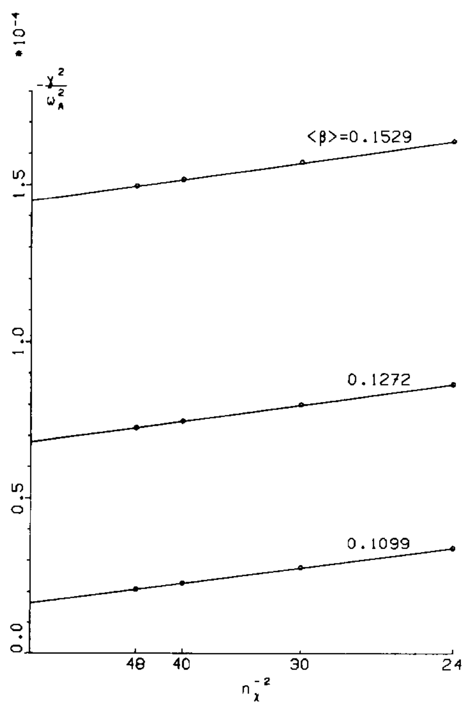

FIG. 6. Extrapolation in HERA to zero mesh size of the eigenvalue of the $m = 1$, $n = 0$ mode occurring in class 1. The number of intervals in the coarsest grid is given by $N_\psi/N_\chi = 24/24$.

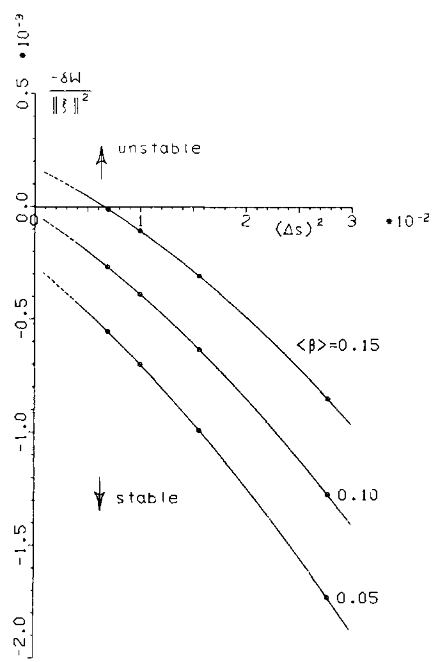

FIG. 7. Extrapolation in BETA to zero mesh size of the eigenvalues of the $m = 1$, $n = 0$ modes occurring in class 1. The number of intervals in the coarsest grid is given by $NS/NU/NV = 6/12/12$.

Also to be noted are the close coincidence of the normalized eigenvalues, which are typically $10^{-2}\omega_A$, corresponding to growth times of $10\,\mu$sec in a fusion plasma ($B = 5$ T, $n = 10^{20}\ m^{-3}$, plasma radius 1 m).

Since the instability is treated two-dimensionally in HERA and three-dimensionally in BETA, the behaviour of the eigenvalues with extrapolation to zero mesh size is characteristically different. While for the simple mode considered here, the extrapolation in HERA is insignificant (see Fig. 6), extrapolation to zero mesh size is essential in BETA (see Fig. 7): even for $\langle\beta\rangle = 15\%$ the unstable behaviour does not actually occur for numbers of mesh points given by $NS/NU/NV = 10/20/20$. In this particularly simple case actually negative values of δW would have been obtained by further mesh refinement, which, however, is not necessary for obtaining reliable eigenvalues.

3.2. *Results for the $m = 2$, $n = 1$ mode.* The structure of the unstable $m = 2$, $n = 1$ mode occurring in the topological torus of five periods of the class 2 equilibria is shown in Fig. 8. Normalized eigenvalues as obtained with HERA are shown in Fig. 9 and indicate as marginal points of the instability window in rotational transform the values

$$\iota_0 = 0.37 \qquad \text{and} \qquad \iota_0 = 0.58.$$

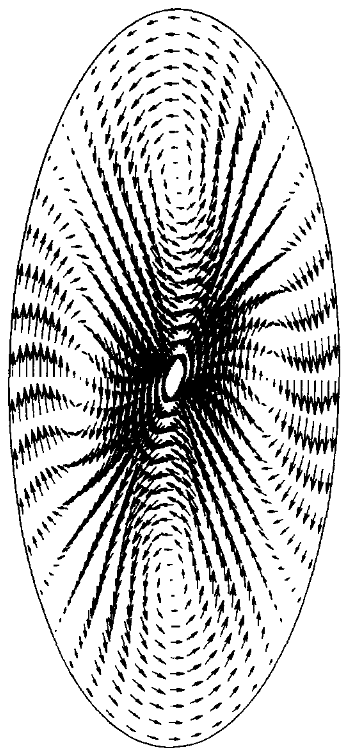

FIG. 8. The fixed-boundary $m = 2$, $n = 1$ mode occurring in class 2.

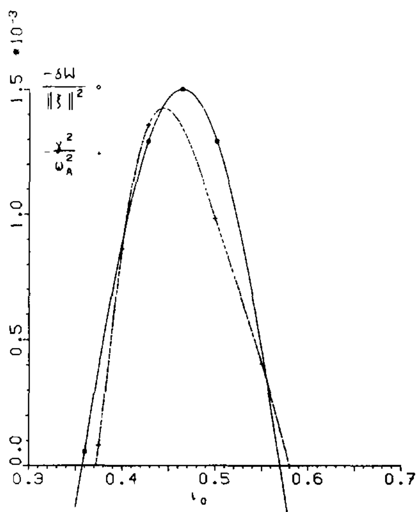

FIG. 9. Normalized extrapolated eigenvalues of the $m = 2$, $n = 1$ mode occurring in class 2 as obtained with BETA and HERA. The grids used in BETA for the extrapolation are given by $6/12/60 \leqslant NS/NU/NV \leqslant 12/24/120$.

The equivalent results obtained with BETA are also shown in Fig. 9, and the marginal points are

$$\iota_0 = 0.36 \quad \text{and} \quad \iota_0 = 0.575.$$

Again, also the eigenvalue curves are essentially equal. Taking into account the shear, we may consider $\iota_E = 0.42$ (see Table II) instead of the above value $\iota_0 = 0.36$ as the smaller marginal point and find that the distance in ι from the resonant value $\iota = 0.5$ has to be about 0.08 for both lower and higher values than the resonant one to avoid the instability. Of course, $\Delta\iota$ will depend on β, which here is $\langle\beta\rangle = 0.03$, and toroidicity, which here vanishes.

4. Concluding Remarks

Comparison of ideal MHD eigenvalues and marginal points in $\langle\beta\rangle$ and ι_0 with respect to the stability of selected modes as obtained from BETA and HERA shows good agreement for the cases considered here. The ease with which these examples are resolved by BETA may serve as an indication of the reliability of this 3D code as applied to truly toroidal cases [6–8]. On the other hand, it is well known that the structure of unstable modes is much more complicated than encountered here in equilibria with significant shear [12], so that for these cases the secondary minimization starting from a test function is must less straightforward. Again, significant tests and hence increase in reliability may be obtained by studying helical equilibria with strong shear [13] in future comparative work.

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